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APPLICATION OF ZERO-RADIUS POTENTIALS TO PROBLEMS OF DIFFRACTION BY SMALL INHOMOGENEITIES IN ELASTIC PLATES†

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A procedure for constructing explicitly solvable models of small inhomogeneities in boundary-contact acoustic problems is presented. The procedure is based on the theory of self-adjoint extensions of symmetric operators and enables the diffraction problem to be reduced to two simpler problems. The first problem is for a totally rigid plate and the second is for an isolated plate. In a number of cases the asymptotic analysis of these problems enables one to construct a model for inhomogeneity in the original boundary-contact problem. This procedure is used to investigate the diffraction of a plane acoustic wave at a plate with a circular aperture of small radius. The problem of diffraction of a plane wave by the aperture in a completely rigid plate and the problem of diffraction of a bending wave by the aperture in an isolated plate can be solved by separation of variables in ellipsoidal and polar coordinates, respectively. The asymptotic behaviour of the field for the original problem in the far zone is obtained.

Zero-radius potentials were introduced by Fermi in the 1930s in order to investigate quantum-mechanical systems. This amounted to specifying "boundary" conditions on the wave function ψ at a point: $(r\psi)^{-1}\partial(r\psi)/\partial|_{r\to 0}\to\alpha$, where r is the distance from the centre of a "potential well", i.e. the point where the zero-radius potential is located, and α is a real number. It was then shown [1] that from a mathematical point of view the setting of a logarithmic derivative defines a self-adjoint extension of some symmetric operator. At the present time zero-radius potentials are frequently used and have become classical in quantum mechanics [2], and also when modelling narrow slits in rigid screens and open resonators [3].

From a physical point of view, an approach based on the application of operator extension theory to the above range of problems enables one to use the smallness of the inhomogeneity to simplify the problem even as one is formulating it. In the same way in which the plate in boundary-contact problems is modelled by an infinitely thin plane, and stiffening ribs by infinitely thin lines (or points in the case of two dimensions), plate inhomogeneities are replaced by point scatterers of special form. To fix the conditions at these scatterers the idea of zero-radius potentials is also employed. The fundamental problem that appears in self-adjoint operator extension theory is the choice of extension parameters that adequately model the object. In the procedure given below this choice is made by splitting the original problem into simpler problems.

Analysis of publications on acoustic boundary-contact problems easily establishes the relation with the associated problems for an isolated, dry structure. (Below these will be called vacuum problems.) Indeed, the extraction of objects corresponding to vacuum problems is performed by the regularization of the integrals and series that formally appear in the application of Fourier integrals and series,‡ and in particular the vacuum objects form the singular terms in the integral equations [4] to which the diffraction problems reduce.

In terms of extension theory this relation between the vacuum and complete problems leads to the possibility of constructing zero-radius potentials from two components. One corresponds to the vacuum problem and describes the influence of the inhomogeneity through its effect on the bending displacements of the plate, and the second corresponds to the diffraction problem for a completely rigid screen and describes the direct effect of the inhomogeneity on the acoustic pressure. Hence, to choose the extension parameters in the complete problem one has to find extension parameters for operators corresponding to the vacuum problem and to the rigid screen problem, and to this end

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one must construct the asymptotic forms of the associated simplified problems. After the extension parameters for the components have been chosen, the two resulting models are united by extension theory methods into a model of a point scatterer in the original boundary-contact problem.

The model described is used to model a circular aperture of small radius in a plate and the diffraction problem is studied. It is clear that in this problem the variables cannot be separated, whereas at the same time in the diffraction problems for the isolated and rigid plates that are investigated in order to obtain the parameters for the potential components, this separation of variables is carried out in ellipsoidal [5] and polar coordinates, respectively.

1. FORMULATION OF THE DIFFRACTION PROBLEM

The problem of the diffraction of an acoustic wave by a thin elastic plate with an inhomogeneity, occupying an arbitrary domain Ω, consists in finding a solution of the Helmholtz equation

$$(\Delta + k^2)\, u(x, y, z) = 0, \quad R^3_+ \backslash \Omega \tag{1.1}$$

with boundary conditions

$$(\Delta^2 - k_0^4)\xi(x, y) + \nu u(x, y, 0) = 0, \quad \xi(x, y) = \partial u / \partial z|_{z=0}, \quad R^2 \setminus (\Omega \cap R^2) \tag{1.2}$$

and certain conditions on $u(x,y,z)$ at $\partial\Omega$ and on $\xi(x,y)$ at $\partial\Omega \cap R^2$. Here u is the acoustic pressure and ξ is a function proportional to the bending displacement of the plate. The conditions at the inhomogeneity fix the mechanical and acoustical conditions and should satisfy the requirements of the existence and uniqueness theorem for solving the scattering problem. These conditions must of necessity be supplemented by the Meixmer conditions and analogous conditions [6] for bending displacements.

The wave process within the system is excited by some incident wave. To fix our ideas we shall assume that the latter is a plane acoustic wave

$$u^{\mathrm{i}} = \exp(i\, kr \cos\vartheta_0 \cos\varphi - i\, kz \sin\vartheta_0) \tag{1.3}$$

The total field in the problem can be represented as the sum of three terms: the incident field u^i, a field u^r that is reflected by the homogeneous plate, and a field u^s that is scattered by the inhomogeneity (so that $u = u^i + u^r + u^s$). The u^r and u^s fields must satisfy the radiation condition.

Since the reflected field u^r is easily found

$$u^{r} = R(\vartheta_0)\exp(i\, kr \cos\vartheta_0\cos\varphi + i\, kz \sin\vartheta_0) \tag{1.4}$$

$$R(\varphi) = (ik(k^4 \cos^4 \vartheta - k_0^4)\sin\vartheta - \nu)\ (ik(k^4 \cos^4 \vartheta - k_0^4)\sin\vartheta + \nu)^{-1}$$

the problem amounts to finding the field u^s. Below we shall construct the asymptotic form of the scattered fields in the far zone, i.e. the asymptotic forms when $R = (x^2 + e^2 + z^2)^{1/2} \to \infty$ and diam $(\Omega) \to 0$.

2. THE OPERATOR STRUCTURE IN THE BOUNDARY-CONTACT PROBLEM

The construction of point models for problems of diffraction by a thin elastic plate is based on extension theory for symmetric operators and requires one to change to an operator formulation of the problem. To this end we consider two spaces: an external space $L_{\text{ext}} = L_2(R^3_+)$ and an internal space $L_{\text{int}} = L_2(R^2)$. The first of these is the space of acoustic pressures ($u(x, y, z) \in L_{\text{ext}}$) and the second is the space of bending plate displacements ($\xi(x,y) \in L_{\text{int}}$). In the external space we specify the operator $H_{\text{ext}} = \Delta$ with domain of definition $D(H_{\text{ext}}) = W^2_{2,N}(R^3_+)$ (where the subscript N denotes the Neumann condition at $z = 0$), and in the internal space the operator $H_{\text{int}} = \Delta^2$ with domain of definition $D(H_{\text{int}}) = W_2^4 = W_2^4(R^2)$. The spectral problem for these operators corresponds to the problem of the diffraction of acoustic waves by a completely rigid plate and the problem of propagation of flexural waves in an isolated plate (a plate situated in a vacuum), respectively.

Consider the operator

$$K = \begin{Vmatrix} \kappa (H^0_{\text{ext}})^2 & \mathbb{O} \\ 1(*) & H_{\text{int}} \end{Vmatrix} \tag{2.1}$$

acting in the space $L = L_{\text{ext}} \oplus L_{\text{int}}$ and defined for pairs of functions $(u_{\text{ext}}, u_{\text{int}}) \equiv U$. The scalar product in L is specified to be the sum of the scalar products of the components in the internal and external spaces

$$\langle U, V\rangle_L = \kappa^{-1} \langle u_{\text{ext}}, \upsilon_{\text{ext}}\rangle + \langle u_{\text{int}}, \upsilon_{\text{int}}\rangle$$

The constant κ in (2.1) is introduced in order to equate the spectral parameters of the external and internal problems: $\kappa = k_0^4 k^{-2}$. The operator $1(*)$ acts from L_{ext} into L_{int} according to the formula $l(u_{\text{ext}}) = \sqrt{(\nu)}\, u_{\text{ext}}|_{z=0}$.

It can be shown that the operator K with domain of definitions

$$D(k) = \{U\colon\ u_{\text{ext}} \in W_2^2(\mathbb{R}^3_+),\ \ u_{\text{int}} \in W_2^4(\mathbb{R}^2),\ \ u_{\text{int}} = \nu^{-1/2} \partial u_{\text{ext}} / \partial z|_{z=0}\}$$

is appropriate for the problem of diffraction by a homogeneous elastic plate and is self-adjoint.

Thus the self-adjoint matrix operator for the problem of diffraction by an elastic plate consists of two self-adjoint problems, corresponding to the problems of an absolutely rigid plate and an isolated plate. It is obvious that the replacement of these components by operators corresponding to the problems of diffraction by inhomogeneities in rigid and isolated plates reduces to the operator problem for diffraction by an elastic plate with an inhomogeneity. If instead of the exact diffraction problems one uses model potentials of zero radius, then the corresponding matrix operator will obviously model the phenomenon of diffraction by an inhomogeneity in an elastic plate.

All this reduces to the following procedure. One first constructs self-adjoint extensions of the operators H_{ext} and H_{int} and chooses parameters for these extensions. One then constructs the extension of K and some of the parameters of this extension are identified with the extension parameters of H_{ext} and H_{int}.

3. ZERO-RADIUS POTENTIALS FOR THE EXTERNAL COMPONENT

Zero-radius potentials for the Helmholtz operator in the presence of a totally rigid screen have been investigated in detail [3]. Hence we shall not repeat all the calculations, and only quote the final formulae in the notation adopted here. The solution of the problem of scattering by a potential of zero radius satisfies the equation

$$(\Delta + k^2)\, u = 0,\ \ z > 0, r > 0 \tag{3.1}$$

the Neumann condition

$$\partial u / \partial z|_{z=0} = 0,\ \ r > 0 \tag{3.2}$$

and some condition at the point $(r = 0, z = 0)$.

In order to find the condition at the point we consider the asymptotic form of an arbitrary solution of (3.1), (3.2) in a neighbourhood of the origin of coordinates

$$u \approx c^0/(4\pi R) + f^0 + o(1) \tag{3.3}$$

Then the condition at the point $(r = 0, z = 0)$ is formulated in the form of a relation between the coefficients c^0 and f^0

$$c^0 = A f^0 \tag{3.4}$$

The parameter A in (3.3) takes real values (including $A = \infty$ which denotes $f^0 = 0$). It has been shown

[3] that all self-adjoint expansions in L_2 corresponding to the external problem are parametrized by condition (3.3).

In addition to condition (3.3) other methods of describing self-adjoint expansions exist, but this version is singled-out by the fact that A in (3.3) depends only on the inhomogeneity and depends neither on the incident field nor on the properties of the acoustic medium. In order to choose a parameter A corresponding to a specific inhomogeneity it is necessary to describe the asymptotic form of the field in the far zone for some model problem.

4. ZERO-RADIUS POTENTIALS IN THE INTERNAL SPACE

A biharmonic operator acts in the internal space and its extension theory has been developed in [7]. We will consider the operator $H^0_{\text{int}} = \Delta^2$ defined on functions in $W^4_{2,0}(R^2)$. Here the zero subscript means that the functions vanish at the origin of coordinates together with all their x and y derivatives up to the second order inclusive. The order of the derivatives, whose values can be fixed at a separate point, is governed by the inclusion theorems [8]. It has been established [7] that defect indices of the operator H^0_{int} are equal to (6, 6), and the defect elements are Green's function

$$G(x,\ y,\ \mu)=\frac{i}{8\mu^2}\left\{H_0^{(1)}(\mu r)-H_0^{(1)}(i\mu r)\right\} \tag{4.1}$$

associated with the complex spectral parameter μ^4, together with its derivatives. Because the singularity of G is of the form $r^2\ln r$, one can perform at least a double differentiation without leaving L_2. Thus we have six defect elements G, G_x, G_y, G_{xx}, G_{xy} and G_{yy}. To simplify the formulae we fix the value of the spectral parameter as follows: $\mu = \exp(i\pi/4)$.

The operator $(H^0_{\text{int}})^*$ adjoint to H^0_{int} is defined on functions represented in the form of the expansion

$$\xi(x,\ y)=\sum_{n+j<3} c^{nj}\frac{\partial^{n+j}G(x,\ y)}{\partial x^n\partial y^j}+\chi(x,\ y)\sum_{n+j<3} b^{nj}\frac{x^n y^j)}{n!j!}+\xi_0(x,\ y) \tag{4.2}$$

Here $\xi_0(x, y)$ belongs to the domain of definition of H^0_{int}, and the function $\chi(x, y)$ is a smooth cut-off function such that $\chi(x, y) = 1$ when $x^2 + y^2 < 1$, $\chi(x, y) = 0$ when $x^2 + y^2 > 2$. The first summation in (4.2) forms the singular component ξ_s of the function ξ, and the remaining terms have no singularities and form the regular component of ξ. The action of the operator $(H^0_{\text{int}})^*$ on $\xi(x, y)$ is given by the following formula

$$(H^0_{\text{int}})^*\xi(x,\ y)=\Delta^2\xi_r(x,\ y)-\xi_s(x,\ y) \tag{4.3}$$

i.e. the operator $(H^0_{\text{int}})^*$ acts on the ξ_r components as Δ^2 and on the ξ_s components as the operator of multiplication by μ^4.

The operator K_{int} (the scattering operator for flexural waves on a zero-radius potential in an isolated plate) is the restriction of $(H^0_{\text{int}})^*$ to functions from a set $D(K_{\text{int}})$ such that the boundary form $I(\xi, \eta) = ((H^0_{\text{int}})^*\xi, \eta)-(\xi, (H^0_{\text{int}})^*\eta)$ vanishes on $D(K_{\text{int}})$. It has been shown [7] that

$$I(\xi,\ \eta)=\sum_{i+j<3}\left(b^{ij}(\xi)\bar{c}^{ij}(\eta)-c^{ij}(\xi)\bar{b}^{ij}(\eta)\right) \tag{4.4}$$

In order for the right-hand side of (4.4) to vanish it is necessary to specify a linear relation between the vectors $\mathbf{c}_{\text{int}} = \{c^{00}, c^{10}, c^{01}, c^{20}, c^{11}, c^{02}\}$ and $\mathbf{b}_{\text{int}} = \{b^{00}, b^{10}, b^{01}, b^{20}, b^{11}, b^{02}\}$ formed by the expansion coefficients of (4.2) for ξ and μ

$$\mathbf{c}^{\text{int}} = \tilde{\mathbf{A}}\mathbf{b}^{\text{int}} \tag{4.5}$$

(where $\tilde{\mathbf{A}}$ is an arbitrary Hermitian matrix). To parametrize all self-adjoint extensions of the operator $H^0{}_{\text{int}}$ in the form (4.5) one has to allow the coefficients of $\tilde{\mathbf{A}}$ to become infinite, as was done with A in (3.4).

Condition (4.5) is inconvenient in the later procedure of constructing a matrix zero-radius potential because the matrix $\tilde{\mathbf{A}}$ depends on the incident field and the properties of the plate. To remove this dependence we formulate a condition similar to (4.5) for the coefficients of the asymptotic expansion of functions in $D(K_{\text{int}})$

$$\xi = \frac{c^{00}}{8\pi} r^2 \ln r + \frac{c^{10}}{4\pi} r \ln r \ \cos\varphi + \frac{c^{01}}{4\pi} r \ln r \ \sin\varphi + \frac{c^{20}}{4\pi}(2\ln r + 1 + 2\cos^2\varphi) +$$
$$+ \frac{c^{11}}{4\pi}\sin\varphi\cos\varphi + \frac{c^{02}}{4\pi}(2\ln r + 1 + 2\sin^2\varphi) + f^{00} - f^{10} r\cos\varphi - f^{01} r \sin\varphi +$$
$$+ \frac{f^{20}}{2} r^2 \cos^2\varphi + f^{11} r^2 \cos\varphi\sin\varphi + \frac{f^{02}}{2} r^2 \sin^2\varphi + o(r^2), \ \ r \to 0 \tag{4.6}$$

The coefficients c^{nj} in (4.5) are identical with the corresponding coefficients in (4.2). The vector $\mathbf{f}_{\text{int}} = \{f^{00}, f^{10}, f^{01}, f^{20}, f^{11}, f^{02}\}$ obviously differs from $\mathbf{b}^{\text{int}}$ by the term $\mathbf{B}(\mu)\mathbf{c}^{\text{int}}$ generated by the asymptotically singular components, with

$$\mathbf{B} = \begin{Vmatrix} i/(8\mu) & 0 & 0 & g & 0 & g \\ 0 & 1/(8\pi) - g & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/(8\pi) - g & 0 & 0 & 0 \\ g & 0 & 0 & 3i\mu/64 & 0 & i\mu/64 \\ 0 & 0 & 0 & 0 & i\mu/32 & 0 \\ g & 0 & 0 & i\mu/64 & 0 & 3i\mu/64 \end{Vmatrix}$$
$$g = [\ln(\mu/2) + \gamma_E - 1 - i\pi/4]/(4\pi)$$

where γ_E is Euler's constant.

The matrix **B** is Hermitian for $\mu = \exp(i\pi/4)$ and, consequently, condition (4.5) can be written in the form

$$\mathbf{c}^{\text{int}} = \mathbf{A}\mathbf{f}^{\text{int}} \tag{4.7}$$

Here the Hermitian matrix **A** no longer depends on the incident field and the properties of the plate far from the inhomogeneity.

5. ZERO-RADIUS POTENTIALS FOR THE OPERATOR K

We will restrict the operator K to vector functions in U such that their external component u_{ext} vanishes at the point (0, 0, 0) and their internal component u_{int} belongs to $W^4_{2,0}(R^2)$, i.e. we make a restriction of the external and internal components similar to that in Sections 3 and 4. The defect elements of the resulting operator K^0 are solutions of the problems

$$\begin{aligned} &-(\kappa\Delta + \lambda)G^0_{\text{ext}}(x, \ y, \ z) = \delta(x)\delta(y)\delta(z) \\ &\sqrt{\nu}G^0_{\text{ext}}(x, \ y, \ 0) + (\Delta^2 - \lambda)G^0_{\text{int}}(x, \ y) = 0 \\ &-(\kappa\Delta + \lambda)G^{nj}_{\text{ext}}(x, \ y, \ z) = 0 \end{aligned} \tag{5.1}$$

$$\sqrt{\nu}G^{nj}_{\text{ext}}(x, \ y, \ 0) + (\Delta^2 - \lambda)G^{nj}_{\text{int}}(x, \ y) = \frac{\partial^n\delta(x)}{\partial^n x}\frac{\partial^j\delta(y)}{\partial^j y} \tag{5.2}$$

The function G^0 is a perturbation of the defect element of the external operator, and the functions G^{nj} are perturbations of the defect elements of the internal operator. Solutions of problems (5.1) and (5.2) can be obtained in the form of Fourier integrals.

$$\begin{aligned} &G^0_{\text{ext}} = -\frac{k^2}{8\pi k_0^4}\int_{-\infty}^{\infty} H_0^{(1)}(\tau r)e^{-m(\tau)z}\frac{\tau(\tau^4 - \lambda)d\tau}{l(\tau)} \\ &G^0_{\text{int}} = \frac{\sqrt{\nu}k^2}{8\pi k_0^4}\int_{-\infty}^{\infty} H_0^{(1)}(\tau r)\frac{\tau d\tau}{l(\tau)} \\ &l(\tau) = \nu - (\tau^4 - \lambda)m(\tau), \ \ m(\tau) = \sqrt{\tau^2 - \lambda k^2 k_0^{-4}} \end{aligned} \tag{5.3}$$

Here the δ-function has been removed from $f = \nu^{-1/2}\partial G^0{}_{ext}/\partial z|_{z=0}$ because the internal component u_{int} is equal to f everywhere except at (0, 0). It is impossible to differentiate G^0 because the external component of $G^0{}_{ext}$ would leave $L_2(R^3{}_+)$. We have

$$G^{nj}_{int} = -\frac{\sqrt{\nu}}{4\pi}\frac{\partial^{n+3}}{\partial x^n \partial y^j}\int_{-\infty}^{\infty} H_0^{(1)}(\tau r) e^{-m(\tau)z}\frac{\tau d\tau}{l(\tau)} \tag{5.4}$$

$$G^{nj}_{int} = -\frac{1}{4\pi}\frac{\partial^{n+j}}{\partial x^n \partial y^j}\int_{-\infty}^{\infty} H_0^{(1)}(\tau r) m(\tau)\frac{\tau d\tau}{l(\tau)}$$

The values of the indices n and j are restricted by the requirement $n + j \leqslant 2$ corresponding to the condition $G^{nj} \in L_2(R^3_+) \oplus L_2(R^2)$. The defect indices of K^0 are therefore equal to (7, 7).

The domain of definition of the adjoint operator $(K^0)^*$ consists of functions representable in the form of the following expansion

$$u = u_0 + \chi\left\{c^0 G^0 + \sum_{n,\,j} c^{nj} G^{nj} + b^0 \left\|\begin{matrix}1\\0\end{matrix}\right\| + \sum_{n,\,j} b^{nj}\frac{x^n y^j}{n!\,j!}\left\|\begin{matrix}0\\1\end{matrix}\right\|\right\},\quad u_0 \in D(K^0) \tag{5.5}$$

Here χ is a smooth cut-off function. Computing the boundary form of the operator $(K^0)^*$, we obtain

$$I(u,\ \upsilon) = \frac{k^2}{2k_0^4}\left(b^0(u)\overline{c^0}(\upsilon) - c^0(u)\overline{b^0}(\upsilon)\right) + \sum_{n,\,j}\left(b^{nj}(u)\overline{c^{nj}}(\upsilon) - c^{nj}(u)\overline{b^{nj}}(\upsilon)\right)$$

It is convenient to introduce the following vectors

$$\mathbf{c} = (k/(k_0^2\sqrt{2})c^0,\quad \mathbf{c}^{int})^{\mathrm{T}},\quad \mathbf{b} = (k/(k_0^2\sqrt{2})b^0,\quad \mathbf{b}^{int})^{\mathrm{T}} \tag{5.6}$$

In terms of **c** and **b** the self-adjointness condition is written in the form

$$\mathbf{b} = \mathbf{M}\mathbf{c} \tag{5.7}$$

where M is an arbitrary Hermitian matrix from $\{C + \infty\}^7$. It depends on all the parameters of the problem, and so condition (5.7) must be rewritten in invariant terms, so that the matrix in the condition depends only on the inhomogeneity (the size and shape of Ω and the nature of the boundary condition). To this end we obtain the asymptotic behaviour of the functions (5.5). From (5.3) and (5.4) it is convenient to select the defect elements of the external and internal operators

$$G^0\ \frac{k^2}{4\pi k_0^4 R}\exp\left\{i\sqrt{\lambda k^2 k_0^{-4}}\,R\right\}\left\|\begin{matrix}1\\0\end{matrix}\right\| + g^0$$

$$G^{nj}\ \frac{\partial^{n+j}}{\partial x^n \partial y^j}\left\{\frac{i}{8\sqrt{\lambda}}\left(H_0^{(1)}(\sqrt{\lambda}r) - H_0^{(1)}(i\sqrt{\lambda}r)\right)\left\|\begin{matrix}0\\1\end{matrix}\right\| + g^{00}\right\}$$

Here g^0 and g^{00} are two-component functions

$$g^0_{ext} = -\frac{\nu k^2}{8\pi k_0^4}\int_{-\infty}^{\infty} H_0^{(1)}(\tau r) e^{-m(\tau)z}\frac{\tau m(\tau)}{l(\tau)}d\tau,\quad g^0_{int} = G^0_{int}$$

$$g^{00}_{ext} = G^{00}_{ext},\quad g^{00}_{int} = \frac{1}{4\pi}\int_{-\infty}^{\infty} H_0^{(1)}(\tau r)\frac{\tau}{l(\tau)}\frac{d\tau}{(\lambda - \tau^4)}$$

Computing asymptotic expansions for g^0 and g^{nj}, one can establish that the singular terms in the asymptotic form of u are given by the defect elements H_{ext} and H_{int}, while the g^0 and g^{00} only introduce corrections to the regular terms. Thus the asymptotic expansion of the internal component u_{int} is a repeat of the expansion (4.6), whereas the expansion of u_{ext} differs from (3.3) by the factor κ

$$u_{\rm ext} = c^0 k^2 / (4\pi k_0^4 R) + \tilde{b}^0 + O(R) \tag{5.8}$$

The asymptotic behaviour of g^0 and $g^{nj} = \partial^{n+j} g^{00} / \partial x^n \partial y^j$ have the form

$$g^0_{\rm ext} = -\frac{\nu k^2}{4\pi k_0^4}\int_0^\infty \frac{\tau d\tau}{m(\tau)l(\tau)} + o(l),\quad g^0_{\rm int} = \frac{k^2}{2k_0^4}\upsilon_0 - \frac{k^2 r^2}{8\pi k_0^4}\upsilon_2 + o(r^2)$$

$$g^{00}_{\rm ext} = \sqrt{\nu}\upsilon_0 + o(l),\quad g^{10}_{\rm ext} = g^{01}_{\rm ext} = g^{11}_{\rm ext} = o(l),\quad g^{20}_{\rm ext} = g^{02}_{\rm ext} = -\frac{\sqrt{\nu}}{2}\upsilon_2 + o(l)$$

$$g^{00}_{\rm int} = I_0 - \frac{I_2}{4}r^2 + o(r^2),\quad g^{10}_{\rm int} = \frac{I_2}{2}x + o(r^2),\quad g^{01}_{\rm int} = \frac{I_2}{2}y + o(r^2) \tag{5.9}$$

$$g^{20}_{\rm int} = -I_2 + \frac{3I_4}{8}x^2 + o(r^2),\quad g^{11}_{\rm int} = \frac{I_4}{4}xy + o(r^2),\quad g^{02}_{\rm int} = -I_2 + \frac{3I_4}{8}y^2 + o(r^2)$$

$$\upsilon_j = \frac{1}{2\pi}\int_0^\infty \frac{\tau^{l+j}}{l(\tau)}d\tau,\quad I_j = -\frac{\nu}{2\pi}\int_0^\infty \frac{\tau^{l+j}}{l(\tau)}(\tau^4 - \lambda)^{-1}d\tau$$

When calculating the asymptotic forms (5.9) we used the property of Hankel functions $H_0^{(1)}(-p) = -H_0^{(2)}(p)$, which enables us to rewrite the integrals for g^0 and g^{00} in the form of integrals along the semi-axis. One can then differentiate the integrands with respect to x and y the required number of times, and then put $r = 0$ and $z = 0$.

Taking into account the asymptotic forms (5.9) we obtain

$$\mathbf{f} = \mathbf{b} + \mathbf{G}\mathbf{c}$$

$$\mathbf{G} = \begin{Vmatrix} B & \mathbf{V} \\ \mathbf{V}' & \mathbf{B}+\mathbf{B}' \end{Vmatrix}$$

$$B = \frac{ik^2}{4\pi k_0^4}\sqrt{\lambda k^2 k_0^{-4}} - \frac{\nu k^2}{4\pi k_0^4}\int_0^\infty \frac{\tau d\tau}{m(\tau)l(\tau)}$$

$$\mathbf{V} = \frac{k}{\sqrt{2k_0^2}}\{\upsilon_0,\ 0,\ 0,\ \upsilon_2,\ 0,\ \upsilon_2\}$$

$$\mathbf{B}' = \begin{Vmatrix} I_0 & 0 & 0 & -I_2 & 0 & I_2 \\ 0 & I_2/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_2/2 & 0 & 0 & 0 \\ -I_2 & 0 & 0 & 3I_4/8 & 0 & I_4/8 \\ 0 & 0 & 0 & 0 & I_4/4 & 0 \\ -I_2 & 0 & 0 & I_4/8 & 0 & 3I_4/8 \end{Vmatrix} \tag{5.10}$$

The matrix $\mathbf{B}$ is defined in Section 4.

If the spectral parameter λ of the problem takes on negative values, the matrix $\mathbf{G}$ in (5.10) is Hermitian and condition (5.7) can be rewritten in the form

$$\mathbf{c} = \mathbf{Z}\mathbf{f};\quad \mathbf{Z} = \mathbf{Z}^* = \begin{Vmatrix} \kappa A & \mathbf{a}^* \\ \mathbf{a} & \mathbf{A} \end{Vmatrix} \tag{5.11}$$

The matrix $\mathbf{Z}$ parametrizes the self-adjoint extensions of the operator K^0. The element A corresponds to the problem of scattering by an inhomogeneity in an absolutely rigid screen, and the matrix $\mathbf{A}$ parametrizes a zero-radius potential in the isolated-plate problem. The vector $\mathbf{a}$ describes the additional interaction between the acoustic pressure and the bending displacements of the plate which appear as a result of the inhomogeneity. The parameter A and matrix $\mathbf{A}$ can be chosen by comparing the asymptotic characteristics of the field in the far zone computed in the rigid-plate and isolated-plate problems, with the characteristics of the field in the far zone computed in the rigid-plate and isolated-plate problems, with the characteristics associated with some zero-radius potential.

6. FORMULATION OF THE PROBLEM OF DIFFRACTION BY A CIRCULAR APERTURE

The problem of the diffraction of an acoustic wave by a thin elastic plate with a circular aperture of small radius consists of finding a solution of the Helmholtz equation (1.1) with boundary conditions

$$(\Delta^2 - k_0^4)\xi(x, y) + \nu u(x, y, 0) = 0, \ \xi(x, y) = \partial u / \partial z|_{z=0}, \ r = \sqrt{x^2 + y^2} > R_0 \tag{6.1}$$

$$u(x, y, 0) = 0, \ r < R_0 \tag{6.2}$$

The mechanical conditions at the edge of the aperture are fixed using the contact conditions

$$M\xi = \left(\frac{\partial^2}{\partial r^2} + \sigma\frac{1}{r}\frac{\partial}{\partial r} + \sigma\frac{1}{r^2}\frac{\partial}{\partial \varphi^2}\right)\xi|_{r=R_0} = 0 \tag{6.3}$$

$$F\xi = \left(\frac{\partial^3}{\partial r^3} + \frac{1}{r}\frac{\partial^2}{\partial r^2} - \frac{1}{r^2}\frac{\partial}{\partial r} + \frac{2-\sigma}{r^2}\frac{\partial^3}{\partial r \partial \varphi^2} - \frac{3+\sigma}{r^3}\frac{\partial^2}{\partial \varphi^2}\right)\xi|_{r=R_0} = 0 \tag{6.4}$$

which specify the absence of bending moments and shear forces on the circle.

The incident wave (1.3) excites a wave process in the system. The reflected field u^r is calculated from formula (1.4). The problem consists of constructing the asymptotic form of the scattered field u^s in the far zone.

7. THE PROBLEM OF DIFFRACTION BY AN ABSOLUTELY RIGID PLATE

We consider the following auxiliary problem of diffractiốn by a circular aperture in a completely rigid screen

$$(\Delta + k^2)\, u = 0, \quad z > 0 \tag{7.1}$$

$$\partial u/\partial z|_{z=0} = 0, \quad r > R_0; \ u|_{z=0} = 0, \ r < R_0$$

The Meixner condition is imposed at the edges of the aperture.

Let the incident field be the plane wave (1.3). The total field u consists of a geometrical part u^g formed by the incident wave u^i and the wave

$$u^r = \exp(ikr\cos\vartheta_0\cos\varphi + ikz\sin\vartheta_0)$$

reflected by a screen without the aperture, and a diffraction correction (the scattered field) u^s. The scattered field should satisfy the radiation condition.

The literature on problem (7.1) is extensive and is reviewed, for example, in [9]. On the one hand, this problem allows of separation of variables in ellipsoidal coordinates and, consequently, its solution can be obtained in the form of infinite series containing elliptic functions [5]. On the other hand, asymptotic approaches have been applied to both the high- and low-frequency cases because of the complicated analysis involved in the exact solution of problem (7.1).

We will use known results [9] for the leading term of the asymptotic form of u^s in the zone $(R \to \infty)$ when $R_0 \ll 1$

$$u^s \approx -2kR_0\exp(ikR)/(\pi kR)$$

This gives the scattering diagram

$$\Psi \approx -2i\pi^{-2}kR_0 \tag{7.2}$$

We then construct the solution of the scattering problem for a zero-radius potential. The scattered field is sought in the form

$$u^s = c^0 \exp(ikR)/(4\pi R)$$

Calculating the asymptotic behaviour

$$u \approx c^0/(4\pi R) + f^0 + o(1), \quad R = \sqrt{x^2+y^2+z^2}$$

and requiring that the condition $c^0 = Af^0$ must be satisfied, we obtain

$$u^s = \frac{A}{1 - ikA/(4\pi)}(u^i(0,\ 0,\ 0) + u^r(0,\ 0,\ 0)\frac{\exp(ikR)}{4\pi R}$$

In order to obtain a radiation pattern coinciding with (7.2) to leading order in R_0, A should equal $-8R_0$.

8. THE PROBLEM OF DIFFRACTION BY AN ISOLATED PLATE

Consider the problem of diffraction of flexural waves by an isolated plate. The flexural oscillations are described by the equation

$$(\Delta^2 - k_0^4)\xi(r,\ \psi) = 0, \quad r > R_0$$

and satisfy boundary conditions (6.1) and (6.3) at the edge of the aperture. Suppose that the field in the plate is excited by the plane incident flexural wave $\xi^i = \exp(ikr\cos\varphi)$.

To construct the scattered field we separate the variables in the polar system of coordinates (r, ψ). We shall seek the scattered field in the form of an expansion

$$\xi^s = \sum_{j=0}^{\infty}\left(\alpha_j H_j^{(1)}(k_0 r) + \beta_j H_j^{(1)}(ik_0 r)\right)\cos j\psi \tag{8.1}$$

Using asymptotic properties of Hankel functions we find

$$\xi^s \approx \sqrt{2\pi/(k_0 r)}\exp(ik_0 r - i\pi/4)\Psi(\varphi) + o\left(1/(k_0 r)\right)$$

which leads to the following formula for the radiation pattern

$$\Psi(\varphi) - \frac{1}{\pi}\sum_{j=0}^{\infty}\alpha_j\cos j\varphi\exp\left(-\frac{i\pi j}{2}\right) \tag{8.2}$$

The radiation pattern is therefore governed by the coefficients α_j of expansion (8.1). To determine these coefficients we substitute (8.1) into boundary conditions (6.1) and (6.2) and compare the expressions for the same functions of the polar angle ψ. After expanding the Hankel functions in terms of the small parameter k_0R_0 we obtain the leading terms of the asymptotic expansions of the coefficients α_0, α_1 and α_3. The remaining coefficients are of lower order in k_0R_0 and their asymptotic forms are not required. We obtain

$$\Psi(\varphi) - \frac{i}{2}\left(\ln\frac{k_0R_0}{2} + \gamma_E - \frac{i\pi}{4} + \frac{\sigma-2}{\sigma(\sigma-1)}\right)^{-1}\cos\varphi + \frac{i}{4}\chi_-(k_0R_0)^2 + o(k_0^2R_0^2)$$

$$\chi_\pm = \frac{\sigma}{1-\sigma} \pm \frac{(1-\sigma)(3-5\sigma)}{5\sigma^2-9} \tag{8.3}$$

We can now choose the matrix **A** parametrizing the zero-radius potential in the isolated-plate problem. To construct this matrix it is necessary to take account of the coordinate-independence of the solution, i.e. in the case when the angle of incidence ξ^1 of the wave is equal to φ_0, in formula (8.3) the angle φ should be replaced by $\varphi - \varphi_0$.

We will seek a solution of the zero-radius potential scattering problem (see Section 4) in the form of a multiple expansion

$$\xi^{(s)} = \sum_{n+j<3} c^{nj} \frac{\partial^{n+j}}{\partial x^n \partial y^j} G(x, y, k_0) \tag{8.4}$$

where Green's function G is given by (4.1). The coefficients c^{nj} are chosen from condition (4.7) on the scatterer. The vector $\mathbf{f}^{\text{int}}$ is given by the incident field ξ^i and the coefficients c^{nj}

$$\mathbf{f}^{\text{int}} = \{\xi^i, -\xi^i_x, -\xi^i_y, \xi^i_{xx}, \xi^i_{xy}, \xi^i_{yy}\}(0, 0) + \mathbf{B}(k_0)\mathbf{c}$$

Calculating the asymptotic behaviour of Green's function as $r \to \infty$ we obtain an expression for the radiation pattern

$$\Psi = \frac{i}{8\pi k_0^2}\{c^{00} + ik_0 c^{10}\sin\varphi + ik_0 c^{01}\cos\varphi - k_0^2 c^{20}\cos^2\varphi -$$

$$-k_0^2 c^{11}\cos\varphi\ \sin\varphi - k_0^2 c^{02}\sin^2\varphi\} \tag{8.5}$$

Comparing expression (8.5) with (8.3), it is easy to determine the matrix $\mathbf{A}$ corresponding to the problem of scattering by a circular aperture in an isolated plate. In calculating the elements of $\mathbf{A}$ it is necessary to require them to be independent of the incident wave, i.e. of the angle of incidence φ_0 and the wave number k_0^4. We obtain

$$a_{22} = a_{33} = 2\left(\ln R_0 + \frac{1}{2} + \frac{(\sigma-2)}{\sigma(1-\sigma)}\right)^{-1},$$

$$a_{44} = a_{66} = \chi_+ R_0^2,\ a_{46} = a_{64} = \chi_- R_0^2$$

The remaining elements are equal to zero.

As was shown above, the matrix $\mathbf{Z}$ parametrizing zero-radius potentials in boundary-contact problems has a partitioned structure. The element A and the matrix $\mathbf{A}$ are given above. The vector $\mathbf{a}$, which describes the additional interaction between the acoustic pressure and the bending displacements of the plate that appear because of the presence of the aperture, is equal to zero because in the classical formulation of the problem such an interaction does not occur in the conditions at the aperture (6.2)–(6.4) (u and ξ never appear in the conditions together). Thus the parameters in the zero-radius model for a circular aperture in an elastic plate are completely determined.

9. SCATTERING BY A ZERO-RADIUS POTENTIAL IN THE ORIGINAL PROBLEM

We will formulate the problem of scattering by a matrix zero-radius potential parametrized by the constructed matrix $\mathbf{Z}$. If there is no inhomogeneity the incident wave u^i (1.3) and the reflected wave u^r (1.4) produce the field u^g. The scattering problem consists of calculating the function u^s which satisfies conditions (1.1) and (1.2) everywhere except at the point (0, 0, 0) and which when summed with the specified function u^g has the asymptotic form (4.6), and $\xi = \partial(u^g + u^s)/\partial z|_{z=0}$ has the asymptotic form (5.8) with coefficients satisfying (5.11) with

$$\mathbf{Z} = \operatorname{diag}(\kappa A, \mathbf{A})$$

The scattered field is sought in the form of the expansion

$$u^s = c^0 G^0_{\text{ext}} + \sum_{n+j<3} c^{nj} G^{nj}_{\text{ext}} \tag{9.1}$$

in which the functions G^0 and G^{nj} (see Section 5) are taken with $\lambda = k_0^4$. It is obvious that representation (9.1) satisfies Eq. (1.1) and condition (6.1). In order for condition (5.11) to be satisfied we use the arbitrariness in the choice of the coefficients c^0 and c^{nj}.

We consider a vector generated by coefficients of the asymptotic expansions u^g and $\partial u^g/\partial z|_{z=0}$

$$\mathbf{d} = (d^0, d^{00}, d^{10}, d^{01}, d^{20}, d^{11}, d^{02})^t$$

where

$$d^0 = 2^{-1/2} k k_0^{-2} \{1 + R(\vartheta_0)\}$$
$$d^{nj} = ik\nu^{-1/2} \sin\vartheta_0 \{R(\vartheta_0) - 1\}(-ik\cos\varphi_0\cos\vartheta_0)^n(-ik\sin\varphi_0\cos\vartheta_0)^j$$

As a result condition (5.11) is rewritten in the form

$$\mathbf{d} + \mathbf{G}\mathbf{c} = \mathrm{diag}(\kappa A, \mathbf{A})\,\mathbf{c} \tag{9.2}$$

where the matrix $\mathbf{G}$ is given by the second formula in (5.10).

Solving system (9.2) we determine the scattered field u^s_{ext}. To construct the asymptotic form of the scattered field in the case of an aperture of small radius ($R_0 \to 0$) in a thin plate we obtain the asymptotic form of the matrix $\mathbf{G}$. For a thin plate the roots of the dispersion equation

$$l(\tau) = 0 \tag{9.3}$$

are approximately

$$\tau_j = \nu^{1/5}\exp\{(2\pi i/5)\,j\},\quad 0 \leqslant j \leqslant 4 \tag{9.4}$$

Using the asymptotic forms (9.4) we calculate the integrals contained in $\mathbf{G}$. To do this we note that by introducing a new variable of integration $t = (\tau^2 - k^2)^{1/2}$ one can reduce the integrals to the form of integrals of rational functions of polynomials, which can be explicitly expressed in terms of the roots of the dispersion equation. Note that since the dispersion equation reduces to an algebraic equation of degree 5, its roots, and consequently, the integrals, can only be computed asymptotically. We finally obtain

$$G_{00} = -\frac{k^2\nu^{1/5}\operatorname{ctg}(\pi/5)}{20k_0^4},\quad G_{00} = G_{01} = -\frac{k\operatorname{tg}(\pi/10)}{10\sqrt{2}k_0^2\nu^{1/10}}$$

$$G_{40} = G_{04} = G_{60} = G_{06} = -\frac{k\operatorname{ctg}(\pi/10)}{20\sqrt{2}k_0^2\nu^{3/10}},\quad G_{11} = \frac{\nu^{-2/5}}{10\cos(\pi/10)}$$

$$G_{14} = G_{41} = G_{16} = G_{61} = \frac{\ln(\nu^{1/5}/2) + \gamma_E - 1}{4\pi}$$

$$G_{22} = G_{33} = \frac{1/2 - \ln(\nu^{1/5}/2) - \gamma_E}{4\pi}$$

$$G_{44} = \frac{2}{3}G_{55} = G_{66} = 3G_{46} = 3G_{6\cdot} = -\frac{3\nu^{2/5}}{30\cos(\pi/10)}$$

Solving system (9.2) we find the asymptotic forms of the expansion coefficients of (9.1)

$$c^0 \approx i\frac{16k_0^8}{k\nu}R_0\mu_0\sin\vartheta_0,\quad c^{00} = c^{01} = c^{11} = 0,\quad c^{10} \approx -\frac{8\pi k^2}{\nu^{1/2}}\mu_1\sin\vartheta_0\cos\vartheta_0$$

$$c^{20} \approx 4\pi i\frac{k^3R_0^2}{\nu^{1/2}}\sigma(\chi_+ - 2\chi_-)\sin\vartheta_0\cos^2\vartheta_0,\quad c^{02} \approx -4\pi i\frac{k^3R_0^2}{\nu^{1/2}}\sigma\chi_-\sin\vartheta_0\cos^2\vartheta_0$$

$$\mu_0 = 1 + \frac{2}{5}\nu^{1/5}R_0\operatorname{ctg}\frac{\pi}{5},\quad \mu_1 = \left(\ln\left(\frac{1}{2}\nu^{1/5}R_0\right) + \gamma_E + \frac{\sigma - 2}{\sigma(1-\sigma)}\right)^{-1}$$

We then find the asymptotic form of the scattered field in the far zone. Computing integrals (5.3) and (5.4) by the stationary-phase method, it can be shown that the scattered field for large R is the sum of spherical waves (the contribution of the stationary-phase point)

$$u_{\text{sph}} \approx \frac{2\pi}{kR}\exp\left(ikR-\frac{i\pi}{2}\right)\Psi_{\text{sph}}(\vartheta_0,\ \varphi_0,\ \vartheta,\ \varphi)$$

and a surface wave (the residue at the pole $\tau = \tau_0$)

$$u_{\text{sur}\,f} \approx \sqrt{\frac{2\pi}{\tau_0 r}}\exp\left(i\tau_0 r-\frac{i\pi}{4}\right)\exp\left(-\sqrt{\tau_0^2-k^2}z\right)\Psi_{\text{sur}\,f}(\vartheta_0,\ \varphi_0,\ \varphi)$$

The radiation patterns of the spherical and surface waves are given by the formulae

$$\Psi_{\text{sph}} \approx \frac{2i}{\pi^2}\left(\frac{k_0^4 k}{\nu}\right)^2 kR_0\mu_0 \sin\vartheta\sin\vartheta_0 + \frac{2}{\pi}\frac{k^6\mu_1}{\nu^{6/5}}\sin\vartheta\sin\vartheta_0\cos\vartheta\cos\vartheta_0\cos\varphi +$$

$$+\frac{k^5}{\pi\nu}(kR_0)^2\sigma\sin\vartheta\sin\vartheta_0\cos^2\vartheta\cos^2\vartheta_0(\chi_+-2\chi_-)\cos^2\varphi-\chi_-\sin^2\varphi) \tag{9.5}$$

$$\Psi_{\text{sur}\,f} \approx -\frac{4}{5\pi}\frac{k_0^4}{\nu^{4/5}}kR_0\mu_0\sin\vartheta_0-\frac{4}{5}\frac{k^2}{\nu^{2/5}}\mu_1\sin\vartheta_0\cos\vartheta_0\cos\varphi -$$

$$-\frac{2}{5}\frac{k}{\nu^{1/5}}(kR_0)^2\sigma\sin\vartheta_0\cos^2\vartheta\cos^2\vartheta_0((\chi_+-2\chi_-)\cos^2\varphi-\chi_-\sin^2\varphi) \tag{9.6}$$

When deriving formulae (9.5) and (9.6) we used the fact that the plate is thin, which enabled us to use the asymptotic form of the root r_0 of the dispersion equation. The formulae derived are also asymptotic with respect to the small aperture radius, because modelling the aperture by a zero-radius potential is only valid when $R_0 \ll 1$.

The pattern Ψ_{sph} is symmetric with respect to the angle of incidence and observation, and both diagrams satisfy the optical theorem.† These properties follow from the self-adjointness of the operator and show that the point model for a small aperture is mathematically well-posed.

The proof for the asymptotic expansions of the diagrams obtained for the spherical and surface waves is not given in this paper, i.e. the validity of the zero-radius models is not proved. However, the procedure for constructing a composite model out of models for the external and internal components can be justified in a number of cases by comparing the field in the model problem with the asymptotic forms constructed by classical methods. This comparison, performed for the case of a short rectilinear crack in an elastic plate, demonstrated the validity of the method.

10. CONCLUSION

Our approach to the construction of leading terms in the asymptotic expansion of a field scattered by an inhomogeneity has enabled us to find the asymptotic forms of the field in a rather complicated boundary-contact problem without having to perform difficult calculations. To obtain these asymptotic forms it was only necessary to find the extension parameters in two auxiliary problems. In the case under consideration these problems were explicitly solvable, and their asymptotic investigation turned out to be elementary. In more complex situations, such as for an aperture of irregular form, the variables in the external and internal problems are not separable, but the well-developed method of matched asymptotic expansions can be used to deal with these problems [11].

An important property of the model is that the number A and the matrix $\mathbf{A}$ depend only on the radius of the aperture and do not depend on the properties of the acoustic medium and the plate. The medium and plate properties are taken into account through a Green's function problem without the

†ANDRONOV, I. V., Low-frequency asymptotic forms in boundary-contact problems in mathematical physics and symmetric-operator extension theory. Candidate dissertation, Leningrad State University.

inhomogeneity. All this indicates that the zero-radius model should be valid in the case of a smoothly inhomogeneous acoustic medium and also in the case of a curved plate.

In the above the operator extension was performed without leaving L_2. To obtain more exact models described by matrices **Z** of higher dimensions which would enable one to obtain a larger number of terms in the asymptotic expansions of the fields, the procedure can be extended to the case of operators extended in spaces larger than L_2.

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